

# A New Version of a Posteriori Choosing Regularization Parameter in Ill-Posed Problems

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**Abstract.** The new version of a posteriori choice (NVAC) of the regularization parameter  $\alpha$  in the classical Tikhonov regularization method is considered. Lemmas and theorems on the error and the asymptotic convergence rate of the regularized solution are proved. A numerical example is given.

**Key words.** The classical Tikhonov regularization method; Choice of the regularization parameter  $\alpha$ ; Estimates for  $\alpha$  and for the regularized solution error.

**AMS classification.** 45B05, 65J20, 65R30.

## 1. Introduction

Consider an operator equation of the first kind

$$Ay = f, \quad y \in H_1, \quad f \in H_2, \quad (1)$$

where  $H_1$  and  $H_2$  are Hilbert spaces and  $A : H_1 \rightarrow H_2$  is a linear bounded operator. Suppose that the exact solution  $\bar{y}$  is the normal pseudosolution [1, 2]. Let, instead of the exact  $f$  and  $A$ , we have  $\tilde{f}$  and  $\tilde{A}$  such that  $\|\tilde{f} - f\| \leq \delta$ ,  $\delta > 0$ ,  $\|\tilde{A} - A\| \leq \theta$ ,  $\theta \geq 0$ . Denote by  $\gamma \equiv (\delta, \theta)$ . Given  $\tilde{f}$ ,  $\tilde{A}$ ,  $\delta$ , and  $\theta$ , the problem is to find an element  $y_\gamma \in H_1$  that is a stable approximation of  $\bar{y}$  such that  $\|y_\gamma - \bar{y}\| \rightarrow 0$  as  $\gamma \rightarrow 0$ .

In the classical Tikhonov regularization method (using stabilizers of the type  $\|y\|_{L_2}^2$  or  $\|y\|_{W_2^p}^2$ ), one solves the equation [1–10]

$$\alpha y_\alpha + \tilde{A}^* \tilde{A} y_\alpha = \tilde{A}^* \tilde{f}, \quad (2)$$

where  $\alpha > 0$  is the regularization parameter.

Well-known *ways for choosing the regularization parameter*  $\alpha$  were developed, namely, the discrepancy principle [11], the generalized discrepancy principle (GDP) [7], the modified discrepancy principle (MDP) [12–17], the cross-validation method [18], the iteration stopping rule by discrepancy [5, 6], the local regularising algorithm [19], the adaptive specialized generalized discrepancy principle (SGDP) [1], etc. Estimates of the error  $\|y_\alpha - \bar{y}\|$  for the regularized solution  $y_\alpha$  were obtained, among them, with use of an a priori information about the solution  $\bar{y}$  (the sourcewise representability, etc.) [1–3, 5–9, 11–17, 20, 21].

However, solving a number of model examples shows the following (see [4, 7], et al.). For finite  $\delta$  and  $\theta$ , the principles can overstate the value of  $\alpha$  in comparison with  $\alpha_{\text{opt}}$ . As a result, the error  $\|y_\alpha - \bar{y}\|$  is overstated in comparison with  $\|y_{\alpha_{\text{opt}}} - \bar{y}\|$ , and the solution  $y_\alpha$  becomes more smooth than  $y_{\alpha_{\text{opt}}}$ , and “the fine structure” of the solution  $y_\alpha$  is lost (cf. [22]). Here,  $\alpha_{\text{opt}}$  is the value of  $\alpha$  for which  $\|y_\alpha - \bar{y}\| = \min_{\alpha}$  (the value of  $\alpha_{\text{opt}}$  can be determined without strong a priori suppositions about the solution only in solving model examples). This effect usually appears when the relative errors  $\delta_{\text{rel}}$  and  $\theta_{\text{rel}} \gtrsim 1\%$  [4, p. 283], [7].

The aim of this paper is the further development of the new version of a posteriori choice of  $\alpha$  (NVAC) [2] concentrating attention on the question about closeness of  $\alpha$  to  $\alpha_{\text{opt}}$  and, as a result, of  $\|y_\alpha - \bar{y}\|$  to  $\|y_{\alpha_{\text{opt}}} - \bar{y}\|$ , furthermore, not so much in asymptotics for  $\delta, \theta \rightarrow 0$ , as for finite  $\delta$  and  $\theta$ . In this paper, the modified formulations of the NVAC’s statements are given, moreover, as far as possible without using the sourcewise representability of  $\bar{y}$ . In this case, the solution error estimates for finite  $\delta$ ,  $\theta$ , and  $\alpha$  depend on the exact solution  $\bar{y}$  that is known only in model examples. And in asymptotics (for  $\delta, \theta, \alpha \rightarrow 0$ ), the order of convergence of  $y_\alpha$  to  $\bar{y}$  will be obtained.

**Remark 1.** Since  $\alpha_{\text{opt}}$  and  $y_{\alpha_{\text{opt}}}$  are known only in model examples but are unknown in real problems, so the efficiency of the new version must be verified for model examples.

## 2. The idea of the NVAC

Let us write Eq. (2) in the form

$$\alpha y_\alpha + \tilde{R} y_\alpha = \tilde{F}, \quad (3)$$

where  $\tilde{R} = \tilde{A}^* \tilde{A}$ ,  $\tilde{F} = \tilde{A}^* f$ .

Along with the operator equation (1), consider the Fredholm integral equation of the first kind

$$A y \equiv \int_a^b K(x, s) y(s) ds = f(x), \quad c \leq x \leq d. \quad (4)$$

In the Tikhonov regularization method, instead of Eq. (4), one solves the equation (for  $H_1 = W_2^1$ ,  $H_2 = L_2$ ) [4, p. 24], [23]

$$\alpha [y_\alpha(t) - \tau y_\alpha''(t)] + \int_a^b \tilde{R}(t, s) y_\alpha(s) ds = \tilde{F}(t), \quad a \leq t \leq b, \quad \tau \geq 0, \quad (5)$$

$$y_\alpha'(a) = y_\alpha'(b) = 0,$$

$$\tilde{R}(t, s) = \tilde{R}(s, t) = \int_c^d \tilde{K}(x, t) \tilde{K}(x, s) dx, \quad (6)$$

$$\tilde{F}(t) = \int_c^d \tilde{K}(x, t) \tilde{f}(x) dx. \quad (7)$$

Actually, the original equation in the Tikhonov regularization method is the equation  $\tilde{A}^* \tilde{A} y = \tilde{A}^* f$  rather than  $\tilde{A} y = f$ . Different variants of the

discrepancy principle [3–9, 11–17, 20, 21] use the error  $\delta$  of the right-hand side  $\tilde{f}$ . However, the function  $\tilde{f}(x)$  does not appear explicitly as a right-hand side in the Tikhonov method. The right-hand side is the function  $\tilde{F}(t)$  (see (3) and (5)). The function  $\tilde{f}(x)$  comes under the integral sign in the expression for  $\tilde{F}(t)$  (see (7)), while the integration operation is a smoothing filter with respect to  $\tilde{f}(x)$ . As a result, random errors in  $\tilde{f}(x)$  will be smoothed to a certain extent. In this case, the relative error in  $\tilde{F}(t)$  can become considerably less than the relative error in  $\tilde{f}(x)$  [2].

Concerning the error  $\theta$  of the operator  $\tilde{A}$ , the factual operator in the Tikhonov method is the operator  $\tilde{R} \equiv \tilde{A}^* \tilde{A}$  rather than  $\tilde{A}$ . Therefore, in choosing  $\alpha$  from a discrepancy, it is more appropriately to use the errors of the elements  $\tilde{F}$  and  $\tilde{R}$  rather than  $\delta$  and  $\theta$  (the errors of  $\tilde{f}$  and  $\tilde{A}$ ). However, on deriving asymptotic estimates for  $\alpha$  and for an error of the solution  $y_\alpha$ , one should use the errors of both the elements  $\tilde{F}$  and  $\tilde{R}$  and ones  $\tilde{f}$  and  $\tilde{A}$ .

In the generalized discrepancy principle (GDP) [7],  $\alpha = \alpha_d$  (from discrepancy) is chosen to be a root of the equation  $\|\tilde{A} y_\alpha - \tilde{f}\|^2 = (\delta + \theta \|y_\alpha\|)^2 + \tilde{\mu}^2$ , where  $\tilde{\mu} = \inf_y \|\tilde{A} y - \tilde{f}\|$  is the incompatibility measure of the equation  $\tilde{A} y = \tilde{f}$ .

According to the Kojdecki way [9],  $\alpha$  is a root of the equation

$$\alpha^q \|\tilde{A}^* \tilde{A} y_\alpha - \tilde{A}^* \tilde{f}\| = \beta \|\tilde{A}\| (\delta + \theta \|y_\alpha\|) \quad (8)$$

or, with regard to (2),

$$\alpha^{q+1} \|y_\alpha\| = \beta \|\tilde{A}\| (\delta + \theta \|y_\alpha\|),$$

where  $q \geq 0$  and  $\beta > 0$  are some numbers. One has proved [2] the following lemma.

**Lemma 1.** *The incompatibility measure  $\tilde{\nu} = \inf_y \|\tilde{R} y - \tilde{F}\|$  of the equation  $\tilde{R} y = \tilde{F}$  is equal to zero.*

Now, we formulate again the new version of the a posteriori choice of  $\alpha$  (NVAC), moreover, the results obtained in [2] will be given without proofs. According to the NVAC, with regard to Lemma 1, the regularization parameter  $\alpha$  is chosen to be a root of the equation [2]

$$\alpha^q \|\tilde{R} y_\alpha - \tilde{F}\| = \beta (\Delta + \Theta \|y_\alpha\|), \quad q \geq 0, \quad \beta > 0, \quad (9)$$

or a root of the equivalent equation

$$\alpha^{q+1} \|y_\alpha\| = \beta (\Delta + \Theta \|y_\alpha\|), \quad q \geq 0, \quad \beta > 0, \quad (10)$$

furthermore,  $\|\tilde{F} - F\| \leq \Delta$  and  $\|\tilde{R} - R\| \leq \Theta$ , where  $\Delta = \Delta(\delta, \theta) > 0$  is an upper estimate for the error of the right-hand side  $\tilde{F}$  and  $\Theta = \Theta(\theta) \geq 0$  is an upper estimate for the error of the operator  $\tilde{R}$ . Denote by  $\Gamma \equiv (\Delta, \Theta)$  and by  $\alpha_n$  a root of (9) or (10) (the symbol “n” denotes “new”).

**Remark 2.** Equation (9) is rather like the equation (8). However, these equations have the difference of principle, namely, in Eq. (8), the errors  $\delta$  and  $\theta$  are used and the factor  $\|\tilde{A}\|$  is separated from  $\delta$  and  $\theta$ , whereas in Eq. (9),  $\Delta$  and  $\Theta$  are used. Meanwhile, the value of  $\|\tilde{A}\|(\delta + \theta \|y_\alpha\|)$  can be considerably

greater than  $\Delta + \Theta \|y_\alpha\|$ . This difference can lead to overstated values of  $\alpha$  and  $\|y_\alpha - \bar{y}\|$ .

### 3. Justification of the New Version of a Posteriori Choosing $\alpha$

Denote the left-hand side of (9) or (10) as

$$\psi(\alpha) \equiv \alpha^q \|\tilde{R} y_\alpha - \tilde{F}\| = \alpha^{q+1} \|y_\alpha\|$$

and the right-hand side of (9) or (10) as

$$\xi(\alpha) \equiv \beta(\Delta + \Theta \|y_\alpha\|).$$

Then Eq. (9) or (10) can be written in the form of the equation

$$\psi(\alpha) = \xi(\alpha). \quad (11)$$

**Lemma 2** [2]. *Under the condition*

$$\begin{aligned} \|\tilde{F}\| &> \beta \Delta, & q = 0, \\ \|\tilde{F}\| &> 0, & q > 0 \end{aligned} \quad (12)$$

*the function  $\psi(\alpha)$  is continuous and strictly monotonically increasing, moreover,*

$$\lim_{\alpha \rightarrow 0+} \psi(\alpha) = 0,$$

$$\lim_{\alpha \rightarrow +\infty} \psi(\alpha) = \begin{cases} \|\tilde{F}\|, & q = 0, \\ 0, & q > 0 \text{ and } \|\tilde{F}\| = 0, \\ \infty, & q > 0 \text{ and } \|\tilde{F}\| > 0, \end{cases}$$

*and function  $\xi(\alpha)$  is continuous and strictly monotonically decreasing, moreover,*

$$\begin{aligned} \lim_{\alpha \rightarrow 0+} \xi(\alpha) &> \beta \Delta > 0, \\ \lim_{\alpha \rightarrow +\infty} \xi(\alpha) &= \beta \Delta > 0. \end{aligned}$$

Now, the NVAC can be formulated as the following theorem.

**Theorem 1.** *Let the equation  $\tilde{A}y = \tilde{f}$ ,  $y \in H_1$ ,  $\tilde{f} \in H_2$ , be solved by the Tikhonov regularization method according to (2) or (3), where  $\|\tilde{f} - f\| \leq \delta$ ,  $\delta > 0$ ,  $\|\tilde{A} - A\| \leq \theta$ ,  $\theta \geq 0$ . Suppose that the regularization parameter  $\alpha$  is chosen to be a root of Eq. (9), (10) or (11), furthermore,  $\|\tilde{F} - F\| \leq \Delta$ ,  $\|\tilde{R} - R\| \leq \Theta$ , where  $\Delta = \Delta(\delta, \theta) > 0$ ,  $\Theta = \Theta(\theta) \geq 0$ . Then, under condition (12), a root  $\alpha = \alpha_n$  of Eq. (11) exists and is unique, and the solution  $y_{\alpha_n}$  can be found by solving Eq. (3) with  $\alpha = \alpha_n$ . If condition (12) is not fulfilled, then  $y_{\alpha_n} = 0$ .*

### 4. Some dependences

Let us establish the dependences  $\Delta = \Delta(\delta, \theta)$  and  $\Theta = \Theta(\theta)$ . The estimate for the error  $\Delta$  of the right-hand side  $\tilde{F}$  has the form [2]

$$\Delta \leq \|\tilde{A}\| \delta + \|\tilde{f}\| \theta, \quad (13)$$

and the estimate for the error  $\Theta$  of the operator  $\tilde{R}$  has the form [2]

$$\Theta \leq 2 \|\tilde{A}\| \theta. \quad (14)$$

**Remark 3.** The estimates (13) and (14) are necessary for justifying the convergence of the NVAC. However, in practice for a finite  $\delta$  and  $\theta$ , the formulas (13) and (14) may give an overstatement of  $\Delta$  and  $\Theta$  (see example in the end of the present paper) and, hence, of  $\alpha_n$  if one uses the upper estimates:  $\Delta = \|\tilde{A}\| \delta + \|\tilde{f}\| \theta$  and  $\Theta = 2 \|\tilde{A}\| \theta$ . This overstatement is caused by that the factor  $\|\tilde{A}\|$  is separated from  $\delta$  and  $\theta$  in the estimates (13) and (14). To obtain more exact estimates of  $\Delta$  and  $\Theta$ , one can use, for example, the algorithms II, III and V from the paper [2].

### 5. Estimates for $\alpha_n$

We give two upper estimates for  $\alpha_n$  in the NVAC. Define [2], [9, p. 78]

$$\alpha_0 = \|\tilde{R}\| = \|\tilde{A}\|^2 = \|\tilde{A}^*\|^2. \quad (15)$$

The condition (12) for  $q = 0$  can be written as

$$\frac{\Delta}{\|\tilde{F}\|} < \frac{1}{\beta}. \quad (16)$$

Let us introduce as an extended variant of condition (16) the following condition [2]

$$\frac{\Delta}{\|\tilde{F}\|} + \frac{\Theta}{\|\tilde{R}\|} \leq \frac{1}{\beta} \frac{\|\tilde{R}\|^q}{2}. \quad (17)$$

Condition (17) can also be considered as a modification of condition (53) in [9]. It is proved [2]

**Lemma 3.** *Under condition (17), one has the inequality*

$$\psi(\alpha_0) \geq \xi(\alpha_0). \quad (18)$$

**Corollary 1** [2]. Since the functions  $\psi(\alpha)$  and  $\xi(\alpha)$  are increasing and decreasing, respectively, relations (15), (17), (18) imply that

$$\alpha_n \leq \alpha_0 = \|\tilde{R}\|. \quad (19)$$

Inequality (19) gives an upper estimate for  $\alpha_n$  in terms of the norm of the operator. It is also proved [2]

**Lemma 4.** *Under condition (12), it holds that*

$$\alpha_n \leq \left[ \beta \left( \frac{2 \|\tilde{R}\|}{\|\tilde{F}\|} \Delta + \Theta \right) \right]^{1/(q+1)}. \quad (20)$$

Inequality (20) gives another upper estimate for  $\alpha_n$  (in terms of the errors in the original data).

**Corollary 2** [2]. Since

$$\frac{2 \|\tilde{R}\|}{\|\tilde{F}\|} \Delta + \Theta \leq \max \left\{ \frac{2 \|\tilde{R}\|}{\|\tilde{F}\|}, 1 \right\} (\Delta + \Theta),$$

the estimate (20) can be written as

$$\alpha_n \leq c_1 (\Delta + \Theta)^{1/(q+1)}, \quad (21)$$

$$c_1 = \left[ \beta \cdot \max \left\{ 2 \|\tilde{R}\|/\|\tilde{F}\|, 1 \right\} \right]^{1/(q+1)} > 0. \quad (22)$$

**Corollary 3** [2]. Inequality (21) generates the asymptotic estimate

$$\alpha_n = O \left( (\Delta + \Theta)^{1/(q+1)} \right), \quad \Delta, \Theta \rightarrow 0. \quad (23)$$

Using (13) and (14), we can write the estimates (21) and (22) also as

$$\alpha_n \leq c_2 (\delta + \theta)^{1/(q+1)}, \quad (24)$$

$$c_2 = \left[ 2\beta \|\tilde{A}\| \cdot \max \left\{ \|\tilde{R}\|/\|\tilde{F}\|, \|\tilde{A}\| \cdot \|\tilde{f}\|/\|\tilde{F}\| + 1 \right\} \right]^{1/(q+1)} > 0, \quad (25)$$

$$\alpha_n = O \left( (\delta + \theta)^{1/(q+1)} \right), \quad \delta, \theta \rightarrow 0. \quad (26)$$

The relations (21), (22), (24), (25) show that the estimate for  $\alpha_n$  decreases with decrease of  $\beta$ .

## 6. Error Estimate for the Regularized Solution

We give a new, more precise, estimate for the error  $\|y_{\alpha_n} - \bar{y}\|$  of the regularized solution  $y_{\alpha_n}$  in the NVAC. In the papers [3, 5, 6, 9, 16, 21] et al., it was shown that in the Tikhonov regularization method there holds the following error estimate for the regularized solution (on the assumption that the exact solution  $\bar{y}$  is sourcewise representable with index 1, i.e.  $\bar{y} = A^* A w$ ,  $w \in H_1$ ):

$$\|y_{\alpha} - \bar{y}\| \leq c_3 \frac{\delta + \theta}{\sqrt{\alpha}} + c_4 \alpha, \quad (27)$$

where  $c_3, c_4 > 0$  are some constants.

Let us use the estimate (27). For  $\alpha_n = O((\delta + \theta)^{1/(q+1)})$  (see (26)) there exist such positive constants  $a_1$  and  $a_2$  that (cf. [9, p. 65])

$$a_1 (\delta + \theta)^{1/(q+1)} < \alpha_n < a_2 (\delta + \theta)^{1/(q+1)}. \quad (28)$$

Hence,

$$\|y_{\alpha_n} - \bar{y}\| \leq \frac{c_3}{\sqrt{a_1}} (\delta + \theta)^{(q+0.5)/(q+1)} + c_4 a_2 (\delta + \theta)^{1/(q+1)}. \quad (29)$$

The estimate (29) makes possible to obtain the following *asymptotic estimates*.

For sufficiently small  $\delta$  and  $\theta$ , we have:

$$\|y_{\alpha_n} - \bar{y}\| \leq c(\delta + \theta)^{\tilde{q}}, \quad c > 0, \quad (30)$$

$$\tilde{q} = \frac{\min\{q + 0.5, 1\}}{q + 1} = \begin{cases} (q + 0.5)/(q + 1), & q \in [0, 0.5], \\ 1/(q + 1), & q \geq 0.5. \end{cases} \quad (31)$$

As  $\delta, \theta \rightarrow 0$ , we obtain the asymptotic estimate for the convergence rate of  $y_{\alpha_n}$  to  $\bar{y}$ :

$$\|y_{\alpha_n} - \bar{y}\| = O\left((\delta + \theta)^{\tilde{q}}\right), \quad (32)$$

as well as (we write again the estimate for  $y_{\alpha_n}$ )

$$\alpha_n = O\left((\delta + \theta)^{1/(q+1)}\right). \quad (33)$$

The best asymptotic estimates are obtained for  $q = 0.5$ :

$$\|y_{\alpha_n} - \bar{y}\| = O\left((\delta + \theta)^{2/3}\right), \quad \alpha_n = O\left((\delta + \theta)^{2/3}\right), \quad (34)$$

i.e. the optimal order of convergence is obtained. This is conform to results of the papers [12–16, 21] et al., in which the optimal order of convergence has also been obtained, but for other ways for choosing  $\alpha$  (the modified discrepancy principle, etc.).

If, e.g.,  $q = 0$  then  $\|y_{\alpha_n} - \bar{y}\| = O\left((\delta + \theta)^{1/2}\right)$  – the suboptimal order of convergence as in the GDP [7].

## 7. Final Theorem

In conclusion, we prove the summarizing theorem.

**Theorem 2.** *Let the equation (2) be solved. Furthermore, the regularization parameter  $\alpha$  is chosen with the help of the NVAC according to (11) by equal  $\alpha = \alpha_n$ . In this case, the estimates (19)–(26) for  $\alpha_n$  and the estimates (29)–(32) for the error  $\|y_{\alpha_n} - \bar{y}\|$  of the regularized solution  $y_{\alpha_n}$  are valid. One has a convergence of the regularized solution  $y_{\alpha_n}$  to the exact solution  $\bar{y}$  as  $\delta, \theta \rightarrow 0$ , i.e. the NVAC generates a regularizing algorithm.*

**Proof.** According to (30), (32),  $\|y_{\alpha_n} - \bar{y}\| \rightarrow 0$  as  $\delta, \theta \rightarrow 0$ . This means that  $y_{\alpha_n} \xrightarrow[\delta, \theta \rightarrow 0]{} \bar{y}$ . Theorem 2 is proved.

## 8. Numerical example

To realize the new version of the a posteriori choice of  $\alpha$ , we have developed the program package NVAC using Fortran PowerStation 4.0. The following *model example* (cf. [10, p. 162]) was solved with the help of this package.

The exact solution was set as a superposition of five gaussians (the solution with variations):

$$\begin{aligned} \bar{y}(s) = & 6.5 e^{-[(s+0.66)/0.085]^2} + 9 e^{-[(s+0.41)/0.075]^2} \\ & + 12 e^{-[(s-0.14)/0.084]^2} + 14 e^{-[(s-0.41)/0.095]^2} + 9 e^{-[(s-0.67)/0.065]^2}, \end{aligned}$$

$a = -0.85$ ,  $b = 0.85$ ,  $c = -1$ ,  $d = 1$ , the kernel

$$K(x, s) = \sqrt{r/\pi} e^{-r(x-s)^2/(1+x^2)},$$

where the exact value  $r$  is  $r = 59.924$ . The numbers of discretization nodes are  $l = 161$  (on  $x$ ) and  $n = 137$  (on  $s$  and  $t$ ). The discretization steps are  $\Delta x = \Delta s = \Delta t = \text{const} = 0.0125$ . In this example,  $\|\bar{y}\| = 7.606$ ,  $\|f\| = 6.907$ ,  $\|A\| = 2.419$ ,  $\|F\| = 7.216$ ,  $\|R\| = 2.196$ . Figure 1 shows the exact solution  $\bar{y}(s)$ , the right-hand side  $f(x)$  (considerably more smooth than  $\bar{y}(s)$ ), and the new right-hand side  $F(t)$  (still more smooth than  $f(x)$ ).

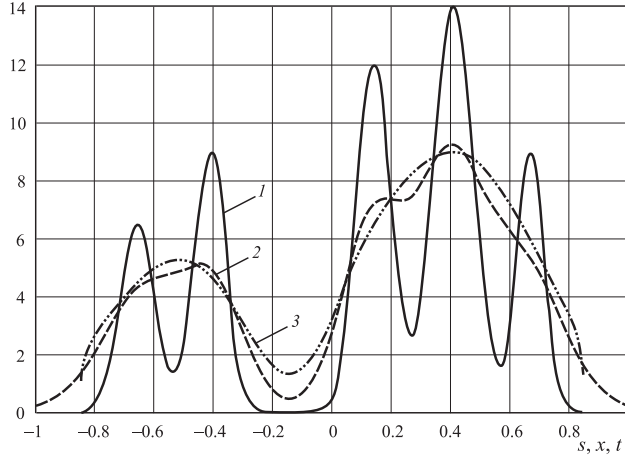


Figure 1: 1 —  $\bar{y}(s)$ ; 2 —  $f(x)$ ; 3 —  $F(t)$

At first, the *direct problem* was solved. The values  $f_i$ ,  $i = 1, \dots, l$ , were calculated. The errors  $\delta f_i$  distributed by the normal law with zero expectation and with the mean square deviation  $\delta = 0.0001$ , 0.15 and 0.5 were added to the values  $f_i$ . The values  $\tilde{r} = 59.920$ , 60 and 65 were used instead of the exact value of  $r$ . Table 1 shows, as an instance, the values of  $\delta$ ,  $\delta/\|f\|$ ,  $\Delta = \|\Delta F\|$ ,  $\Delta/\|F\|$  and (for comparison)  $\|\tilde{A}\|\delta + \|\tilde{f}\|\theta$  for  $\tilde{r} = 60$ . Such value of  $\tilde{r}$  corresponds to the following parameters:  $\theta = \|\Delta A\| = 1.321 \cdot 10^{-3}$ ,  $\theta/\|A\| = 5.46 \cdot 10^{-4} = 0.0546\%$ ,  $\Theta = \|\Delta R\| = 1.194 \cdot 10^{-3}$ ,  $\Theta/\|R\| = 5.44 \cdot 10^{-4} = 0.0544\%$ ,  $2\|\tilde{A}\|\theta = 6.392 \cdot 10^{-3}$ .

Table 1

$\delta$	$\delta/\ f\ $	$\Delta = \ \Delta F\ $	$\Delta/\ F\ $	$\ \tilde{A}\ \delta + \ \tilde{f}\ \theta$
0.0001	$1.448 \cdot 10^{-5}$ $\approx 1.4 \cdot 10^{-3}\%$	$0.6691 \cdot 10^{-3}$	$0.927 \cdot 10^{-4}$ $\approx 0.93 \cdot 10^{-2}\%$	$9.4 \cdot 10^{-3}$
0.15	$2.172 \cdot 10^{-2}$ $\approx 2.2\%$	0.01878	$0.259 \cdot 10^{-2}$ $\approx 0.26\%$	0.3721
0.5	$7.239 \cdot 10^{-2}$ $\approx 7.2\%$	0.06256	$0.867 \cdot 10^{-2}$ $\approx 0.87\%$	1.219

Furthermore, the operator norms  $\|A\|$ ,  $\theta = \|\tilde{A} - A\|$ ,  $\|R\|$ , and  $\Theta = \|\tilde{R} - R\|$



were calculated by means of the Hilbert–Schmidt norm, e.g.,

$$\|A\| = \left\{ \int_a^b \int_c^d K^2(x, s) dx ds \right\}^{1/2}.$$

Comparing the values of  $\Delta$  and  $\|\tilde{A}\|\delta + \|\tilde{f}\|\theta$ , as well as  $\Theta$  and  $2\|\tilde{A}\|\theta$  (see (13) and (14)) we see that the upper estimates  $\|\tilde{A}\|\delta + \|\tilde{f}\|\theta$  and  $2\|\tilde{A}\|\theta$  overstate by one order the values of  $\Delta$  and  $\Theta$ , and comparison of  $\delta/\|f\|$  and  $\Delta/\|F\|$  shows that  $\Delta/\|F\|$  less by one order than  $\delta/\|f\|$  for  $\delta/\|f\| \gtrsim 1\%$ . About this, one says already above.

Afterwards, the *inverse problem* was solved. Equation (5) was solved by the quadrature method at  $\tau = 1$  [4, pp. 249–251]. Figure 2 shows some curves of the relative solution error  $\|y_\alpha - \bar{y}\|/\|\bar{y}\|$  (it can be calculated only in solving a model example with known  $\bar{y}$ ).

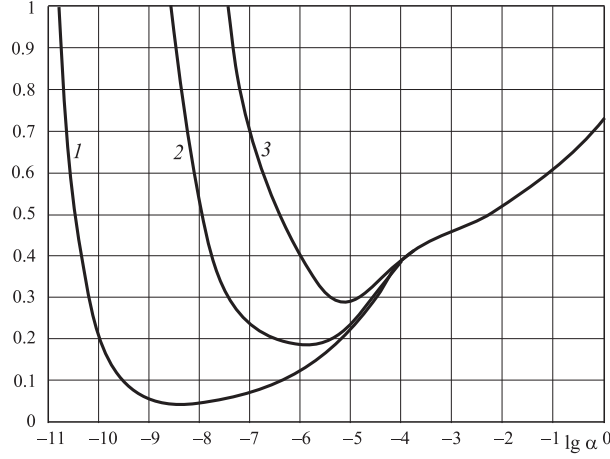


Figure 2: The relative solution error  $\|y_\alpha - \bar{y}\|/\|\bar{y}\|$  at  $\tau = 1$  1 —  $\delta = 0.0001$ ,  $\tilde{r} = 59.920$ ; 2 —  $\delta = 0.15$ ,  $\tilde{r} = 60$ ; 3 —  $\delta = 0.5$ ,  $\tilde{r} = 65$

Table 2 shows, as an instance, the values of  $\alpha_{\text{opt}}$ ,  $\alpha_n$  and the relative errors of the solutions  $y_{\alpha_{\text{opt}}}$  and  $y_{\alpha_n}$  for  $\tilde{r} = 60$ ,  $q = 0$ ,  $\tau = 1$ ,  $\beta = 1$  and  $\beta = 0.1$ .

$\delta$	$\lg \alpha_{\text{opt}}$	$\ y_{\alpha_{\text{opt}}} - \bar{y}\ /\ \bar{y}\ $	$\lg \alpha_n$		$\ y_{\alpha_n} - \bar{y}\ /\ \bar{y}\ $	
			$\beta = 1$	$\beta = 0.1$	$\beta = 1$	$\beta = 0.1$
0.0001	−8.7	0.0385	−5.1	−6.2	0.2107	0.1099
0.15	−5.8	0.1848	−4.3	−5.7	0.3466	0.1858
0.5	−5.2	0.2644	−3.6	−5.2	0.4311	0.2644

Figure 3 shows the logarithms of the functions  $\psi(\alpha) = \alpha^q \|\tilde{R}y_\alpha - \tilde{F}\|$  and  $\xi(\alpha) = \beta(\Delta + \Theta\|y_\alpha\|)$ .

Figure 4 shows the exact solution  $\bar{y}(s)$  and the regularized solutions  $y_\alpha(s)$  at  $\alpha = \alpha_{\text{opt}} = 10^{-5.8}$ ,  $\alpha = \alpha_n = 10^{-5.7}$  ( $\beta = 0.1$ ) and  $\alpha = \alpha_n = 10^{-4.3}$  ( $\beta = 1$ ) for  $\delta = 0.15$ ,  $\tilde{r} = 60$ ,  $q = 0$ ,  $\tau = 1$ .

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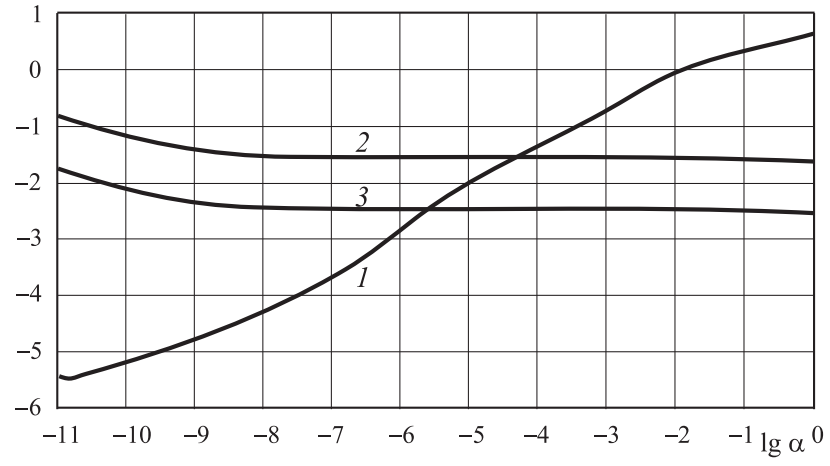


Figure 3: 1 —  $\lg \psi(\alpha)$ ; 2 —  $\lg \xi(\alpha)$ ,  $\beta = 1$ ; 3 —  $\lg \xi(\alpha)$ ,  $\beta = 0.1$

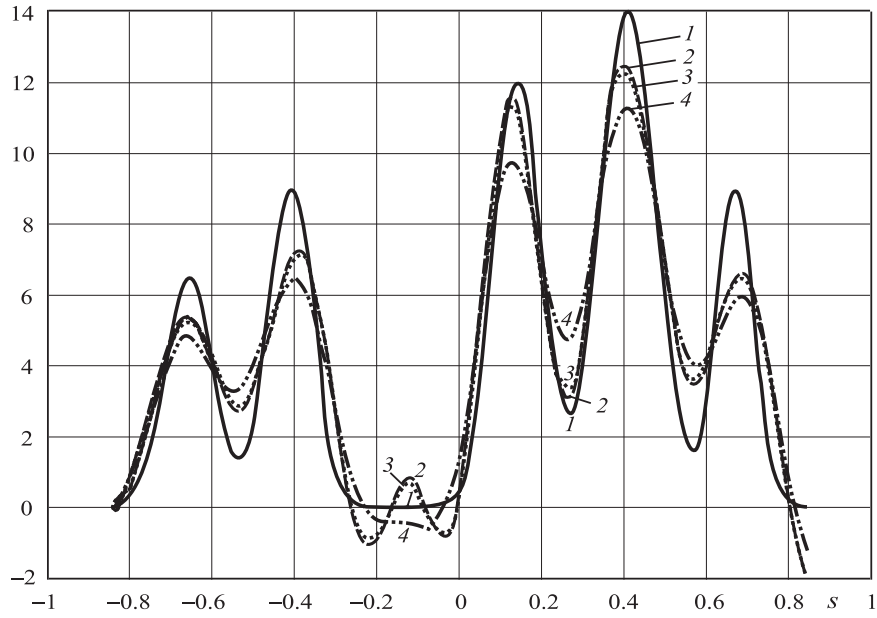


Figure 4: 1 —  $\bar{y}(s)$ ; 2 —  $y_\alpha(s)$ ,  $\alpha = \alpha_{\text{opt}} = 10^{-5.8}$ ; 3 —  $y_\alpha(s)$ ,  $\alpha = \alpha_n = 10^{-5.7}$ ;  $\beta = 0.1$ ; 4 —  $y_\alpha(s)$ ,  $\alpha = \alpha_n = 10^{-4.3}$ ;  $\beta = 1$

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